Towards microscopic formalism of the rotational motion for an odd-A nucleus

Kosai Tanabe *(Saitama Univ., Japan)*

and

Kazuko Sugawara-Tanabe *(RIKEN, Japan)*

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§ 1. Triaxially deformed odd-A nucleus and wobbling motion

-- Application of particle-rotor model --

§ 2. Microscopic description of nuclear states near yrast region

-- Constrained Hartree-Fock-Bogoliubov (CHFB) approximation

and Quantum number-projected HFB approach –

§ 3. Relation between Space-fixed and Body-fixed coordinates

§ 4. Extension of the cranking model to triaxially deformed system

§ 1. Triaxially deformed odd-A nucleus and wobbling motion -- Application of particle-rotor model --

We adopt the particle-rotor Hamiltonian given by

$$H = H_{\rm rot} + H_{\rm sp},$$

with

$$H_{\rm rot} = \sum_{k=x,y,z} A_k (I_k - j_k)^2, \quad A_k = 1/(2\mathcal{J}_k)\ (k = 1,2,3 \text{ or } x,y,z)$$

Scaling factor:

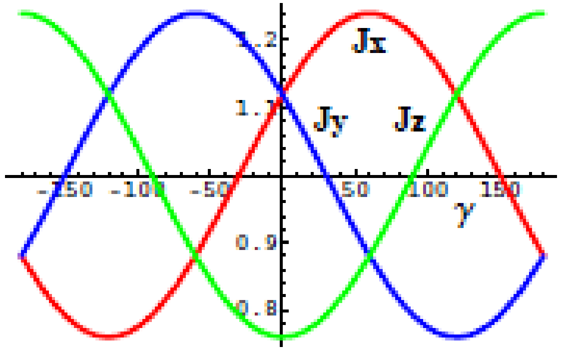
$$H_{\rm sp} = \frac{V}{j(j+1)}[\cos\gamma(3j_z^2 - \vec{j}^2) - \sqrt{3}\sin\gamma(j_x^2 - j_y^2)], \qquad s = \mathcal{J}_0 V$$

$$\left\{\sqrt{\frac{2I+1}{16\pi^2}}[\mathcal{D}^I_{MK'}(\theta_i)\phi^j_{\Omega'} + (-1)^{I-j}\mathcal{D}^I_{M-K'}(\theta_i)\phi^j_{-\Omega'}];\ |K' - \Omega'| = \text{even}, \quad \Omega' > 0\right\},$$

Moment of Inertia (MoI) vs γ (Lund Conv.)

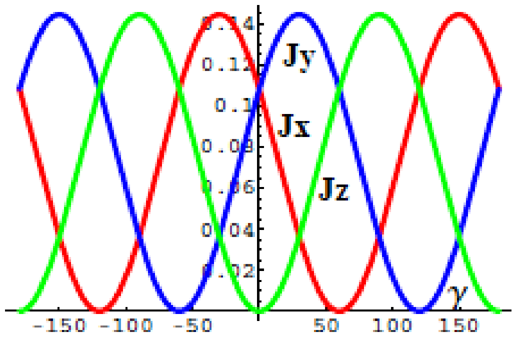
Rigid- body MoI

$$\mathcal{J}_k^{\mathrm{rig}} = \frac{\mathcal{J}_0}{1+(\frac{5}{16\pi})^{1/2}\beta_2}\left[1-\left(\frac{5}{4\pi}\right)^{1/2}\beta_2\cos\left(\gamma+\frac{2}{3}\pi k\right)\right]$$



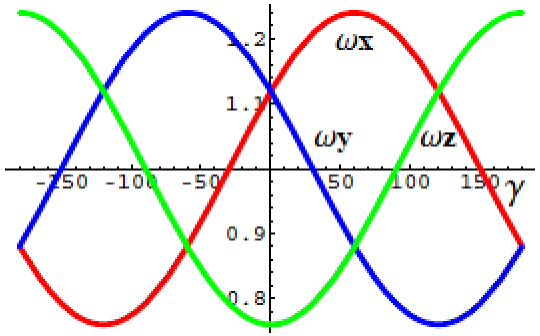
Hydro. MoI

$$\mathcal{J}_k^{\mathrm{hyd}} = \frac{4}{3}\mathcal{J}_0\sin^2\left(\gamma+\frac{2}{3}\pi k\right)$$



Harmonic-oscillator strength

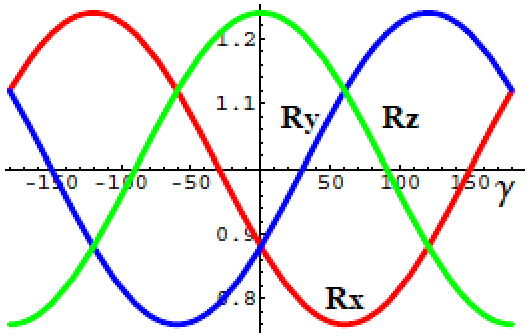
$$\omega_k = \omega_0\left(1-\beta_2\sqrt{\frac{5}{4\pi}}\cos\left(x+\frac{2\pi k}{3}\right)\right)$$



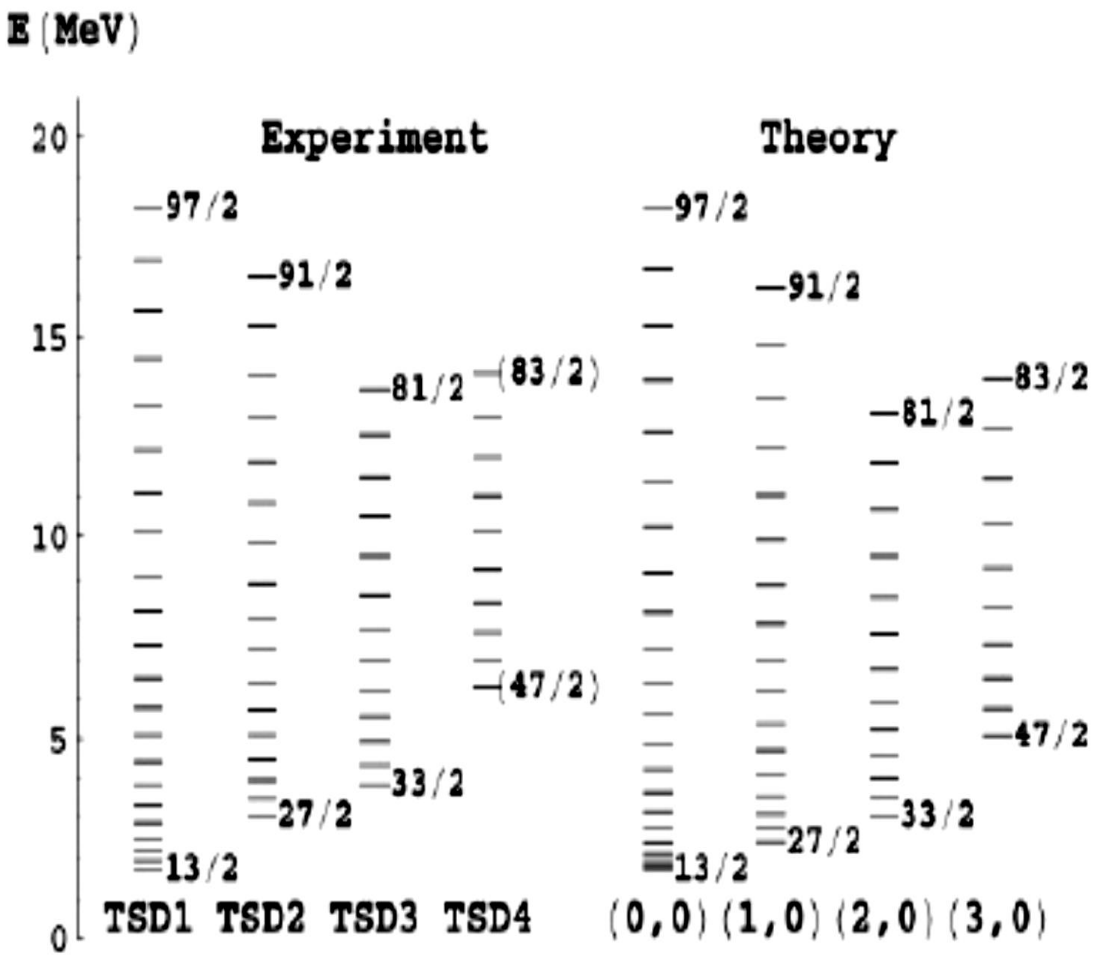
$\beta_2 = 0.38$

Radius

$$R_k = R_0[1+\sqrt{5/(4\pi)}\beta_2\cos(\gamma+2\pi k/3)]$$



^{163}Lu



Experiment:

S. W. Ødegård *et al.*, Phys. Rev. Lett. **86**, 5866 (2001).
D. R. Jensen *et al.*, Phys. Rev. Lett. **89**, 142503 (2002).
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A. Görgen *et al.*, Phys. Rev. C **69**, 031301(R) (2004).
G. Schönwaßer *et al.*, Phys. Lett. **B552**, 9 (2003).
H. Amro *et al.*, Phys. Lett. **B553**, 197 (2003).

Theory:

K.Tanabe and K.Sugawara-Tanabe,
Phys.Rev.C73(2006)034365;
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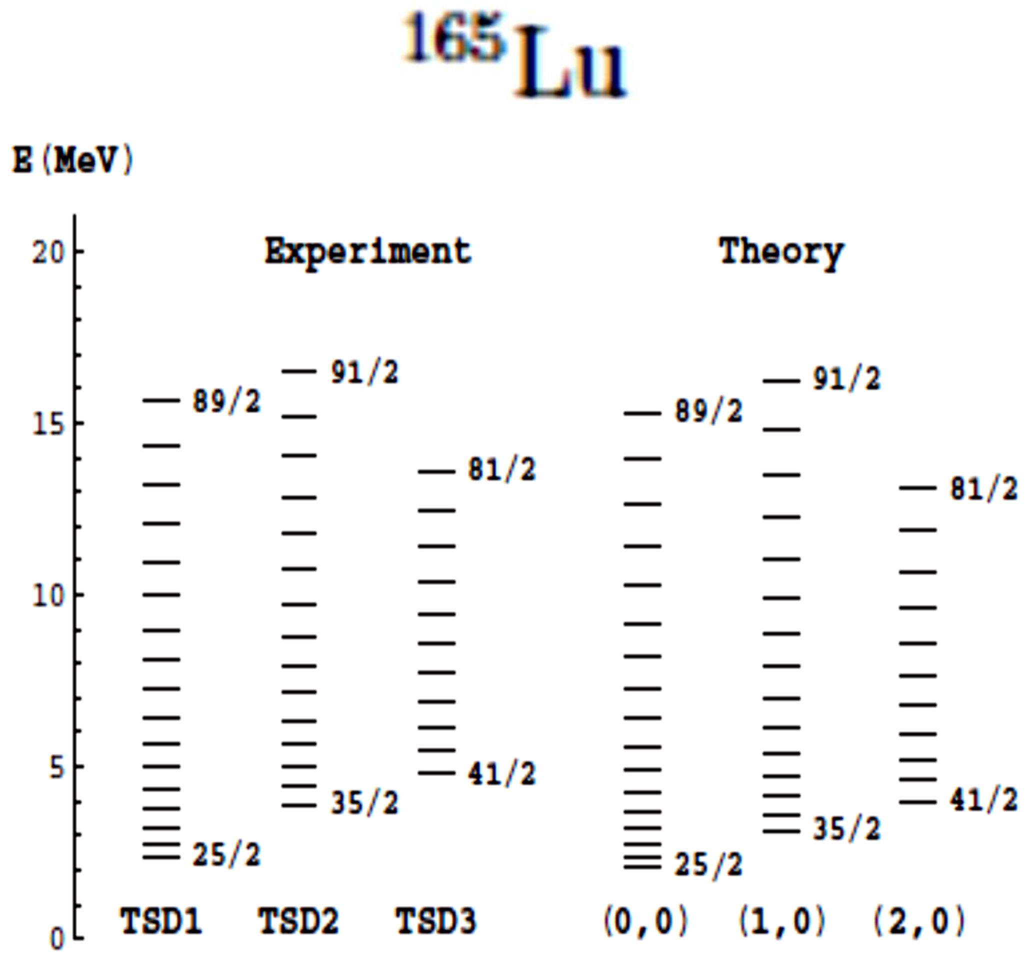
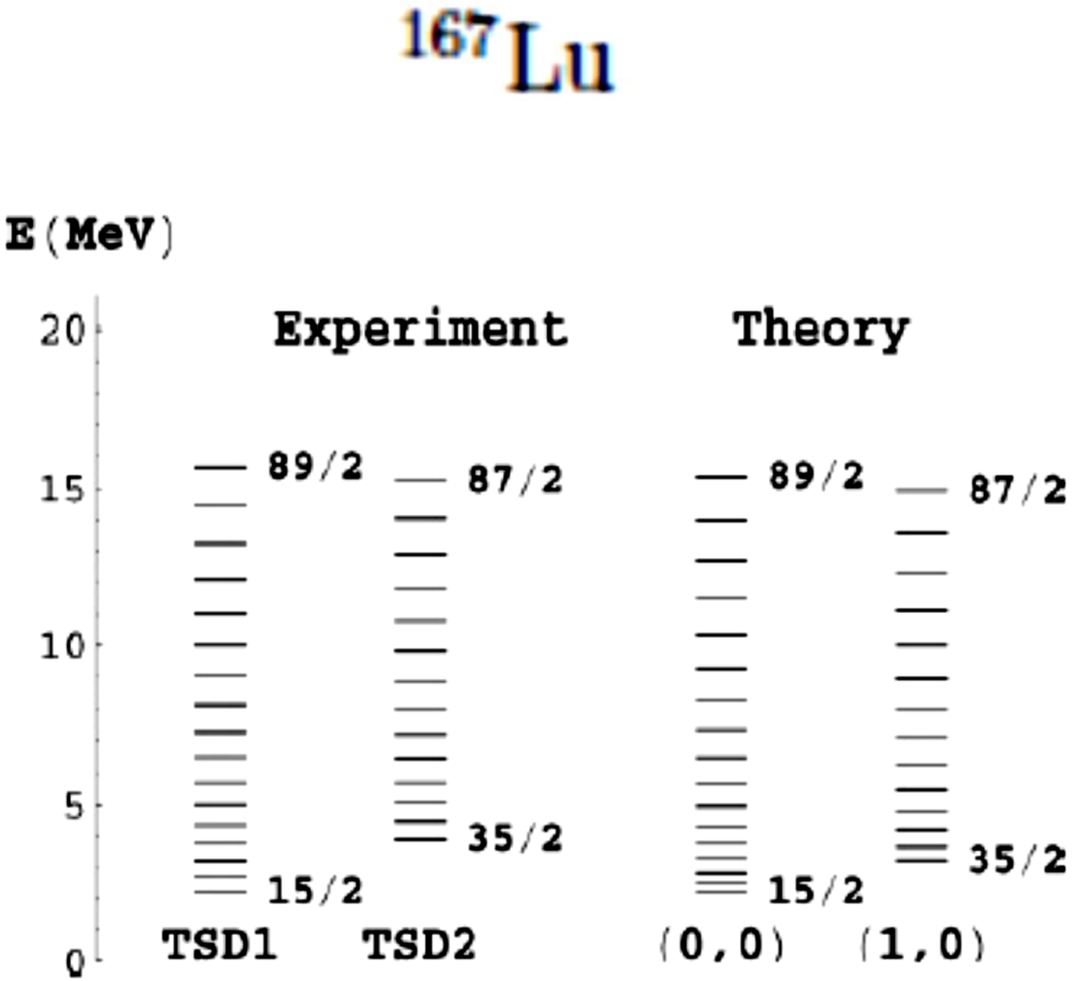
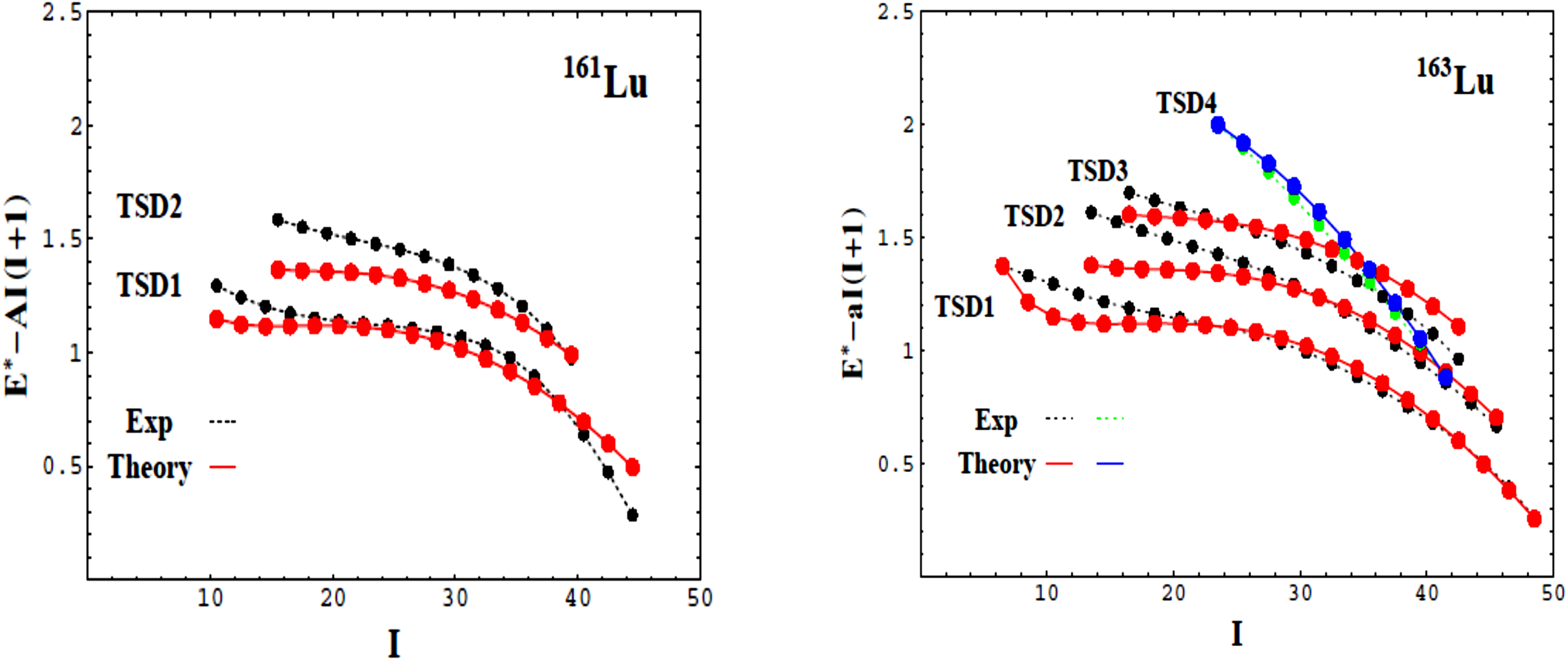


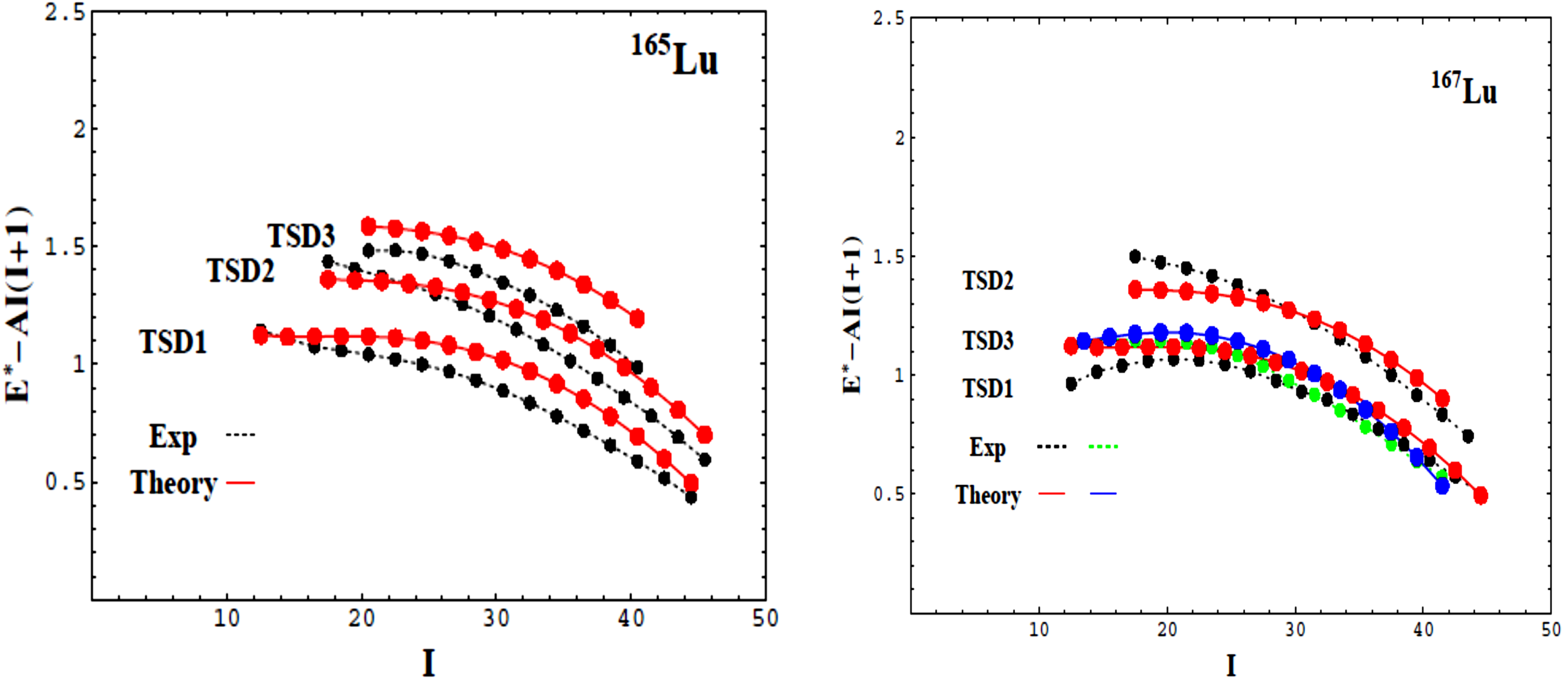
165Lu
E (MeV)
Experiment
Theory
20
15
10
5
0
89/2
91/2
81/2
25/2
35/2
41/2
89/2
91/2
81/2
25/2
35/2
41/2
TSD1
TSD2
TSD3
(0,0)
(1,0)
(2,0)



167Lu
E (MeV)
Experiment
Theory
20
15
10
5
0
89/2
87/2
15/2
35/2
89/2
87/2
15/2
35/2
TSD1
TSD2
(0,0)
(1,0)

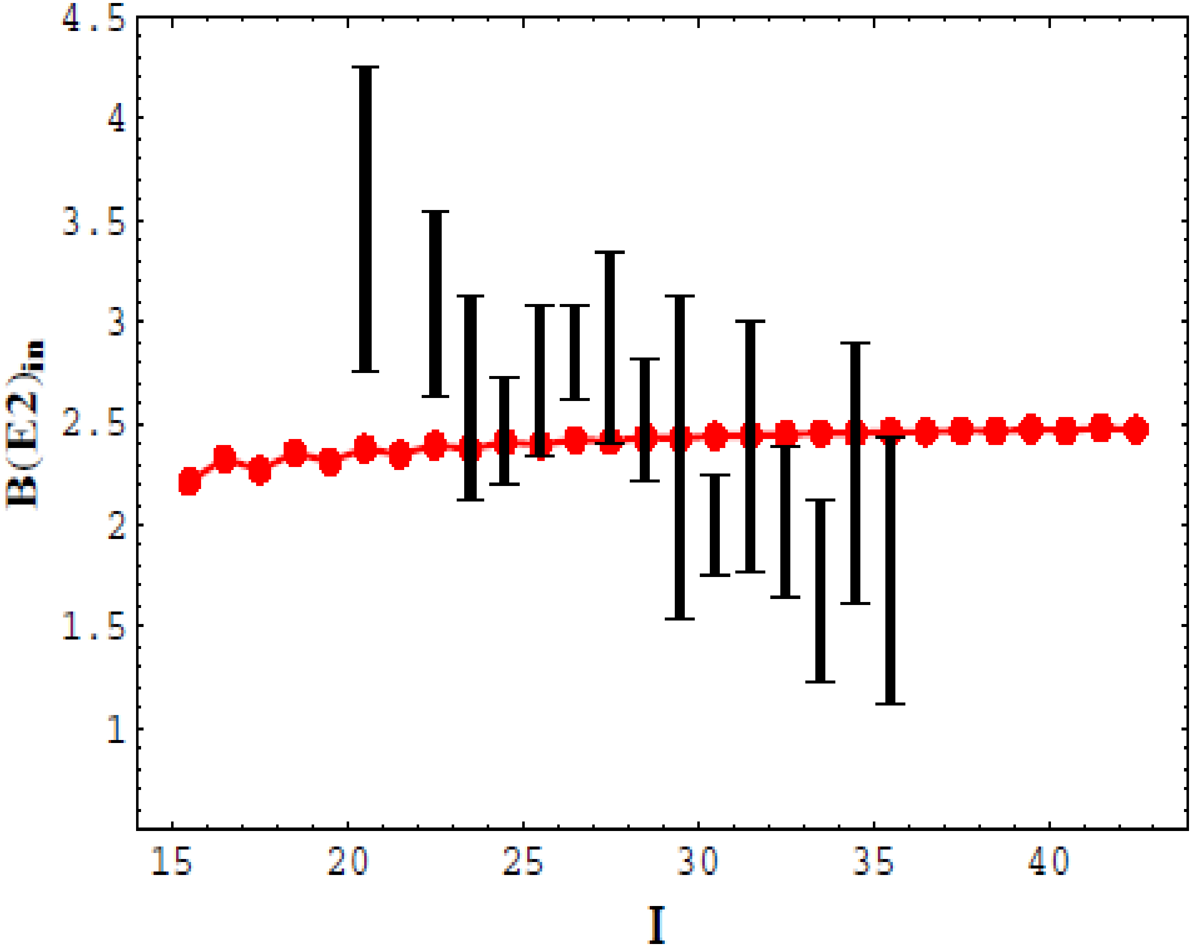


161Lu
TSD2
TSD1
Exp
Theory
E*−AI(I+1)
I
163Lu
TSD4
TSD3
TSD2
TSD1
Exp
Theory
E*−aI(I+1)
I

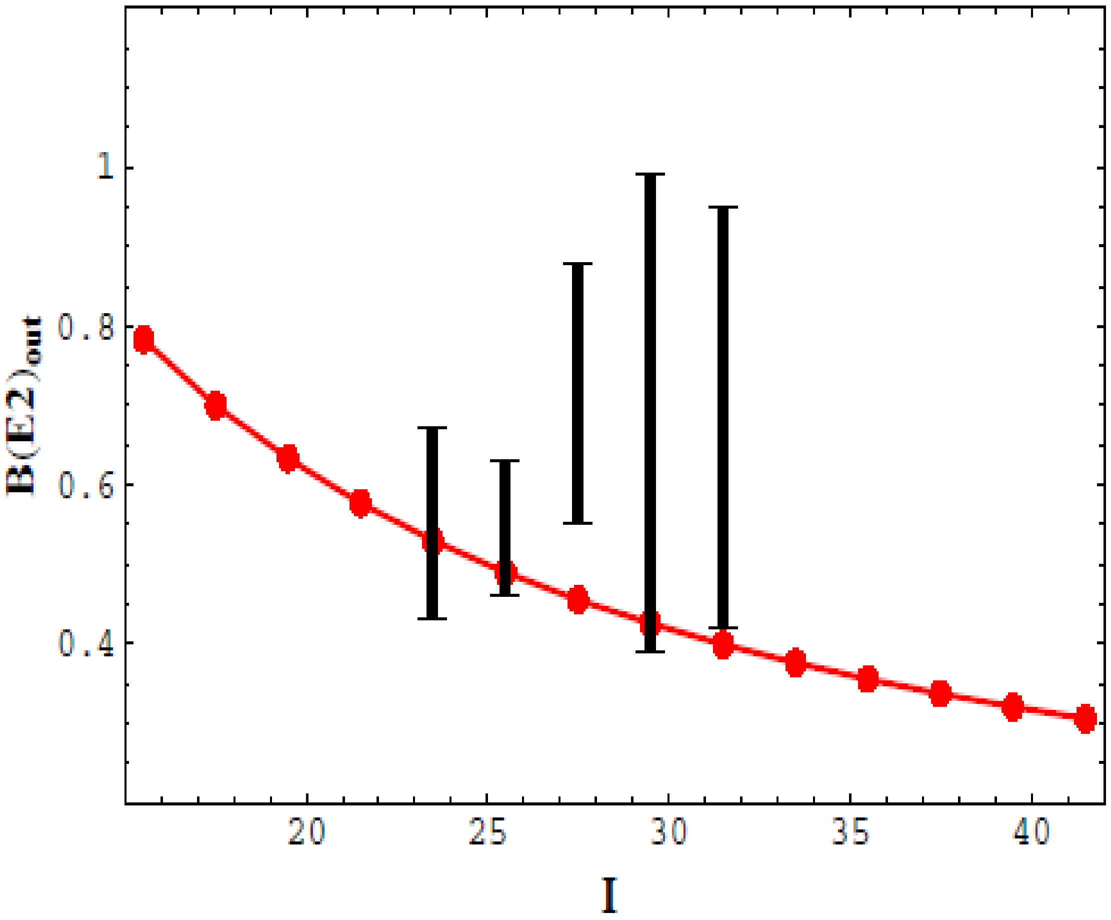


165Lu
TSD3
TSD2
TSD1
Exp
Theory
E*−AI(I+1)
I
167Lu
TSD2
TSD3
TSD1
Exp
Theory
E*−AI(I+1)
I

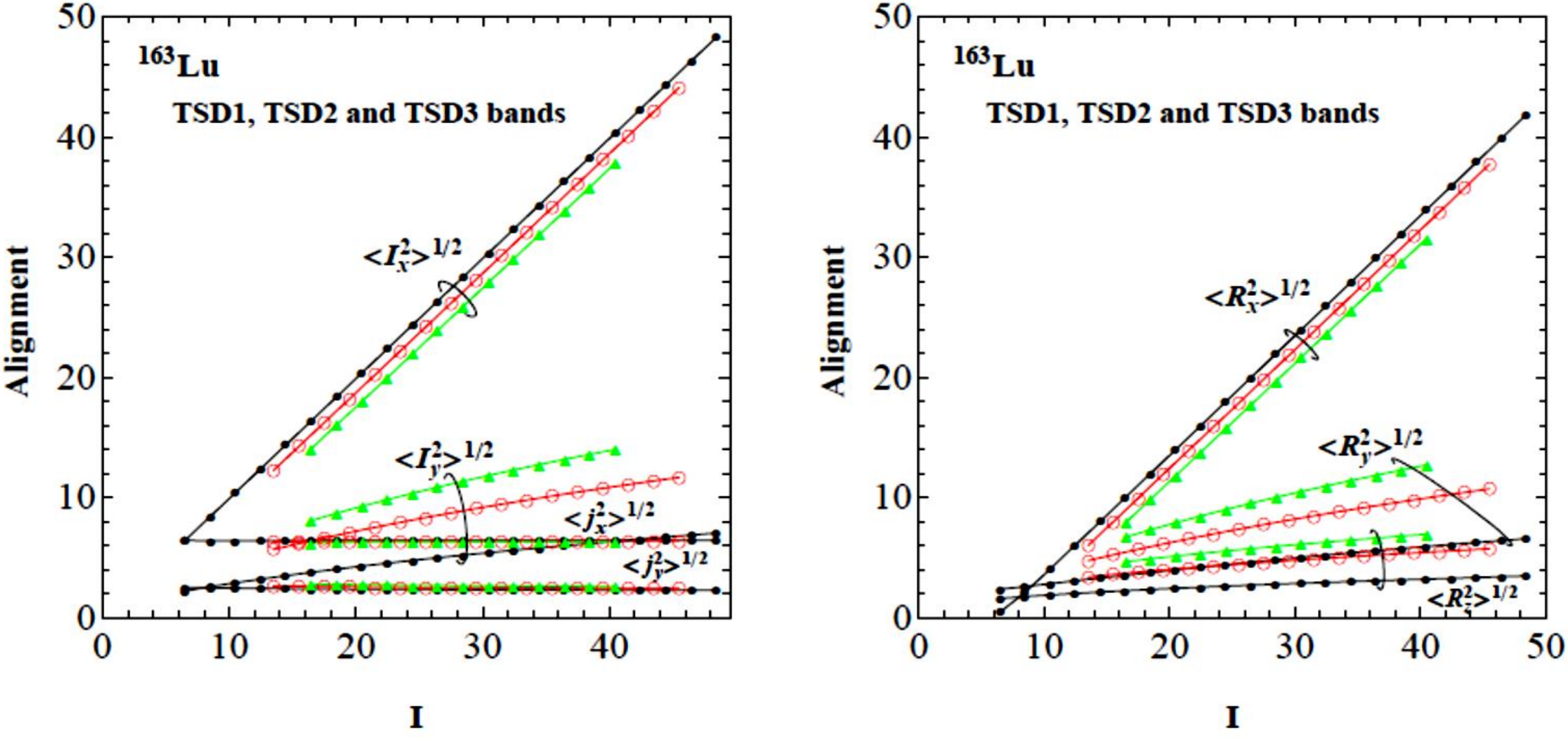
B(E2)in



B(E2)out



Alignment of total a.m. I, rotor a.m. R and single particle j



[9] D. J. Hartley *et al.*, Phys. Rev. C **80**, 041304(R) (2009).

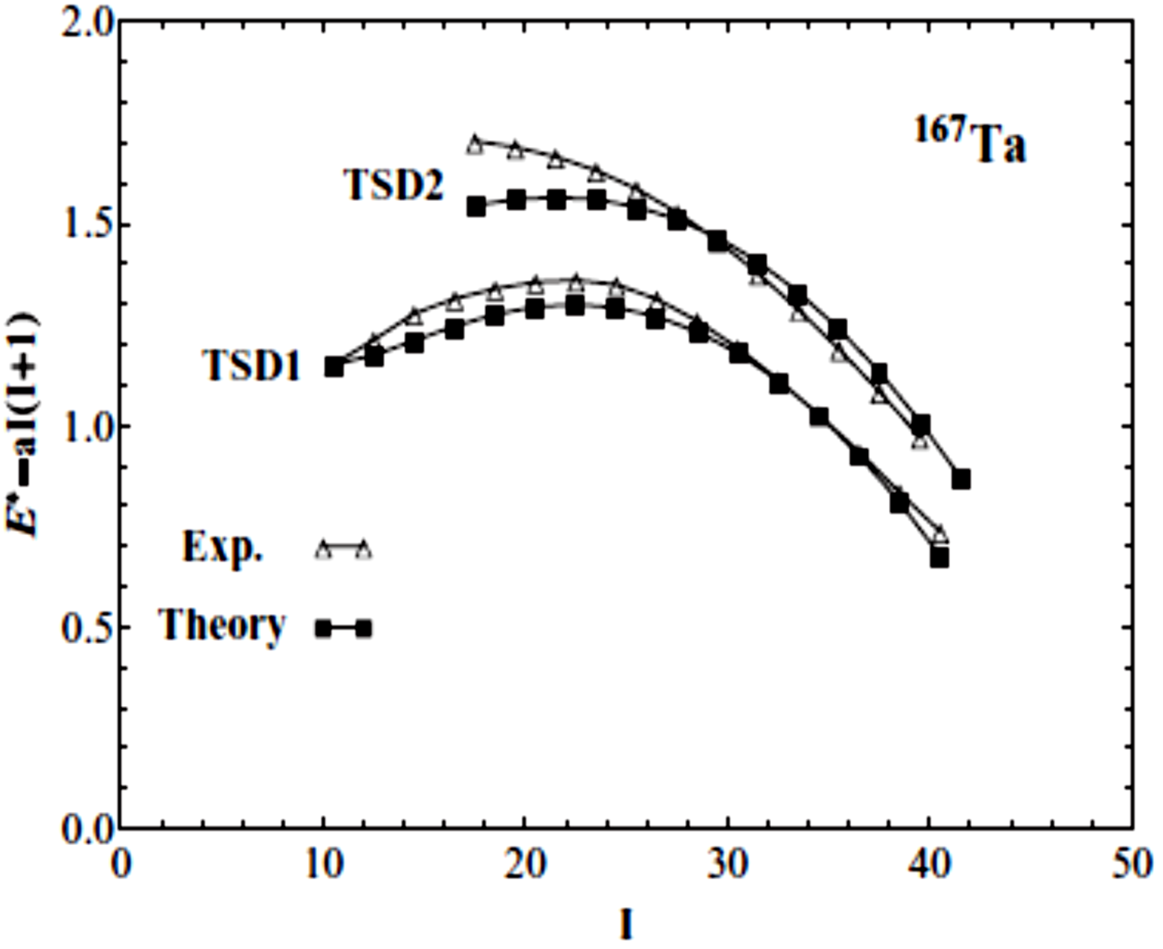
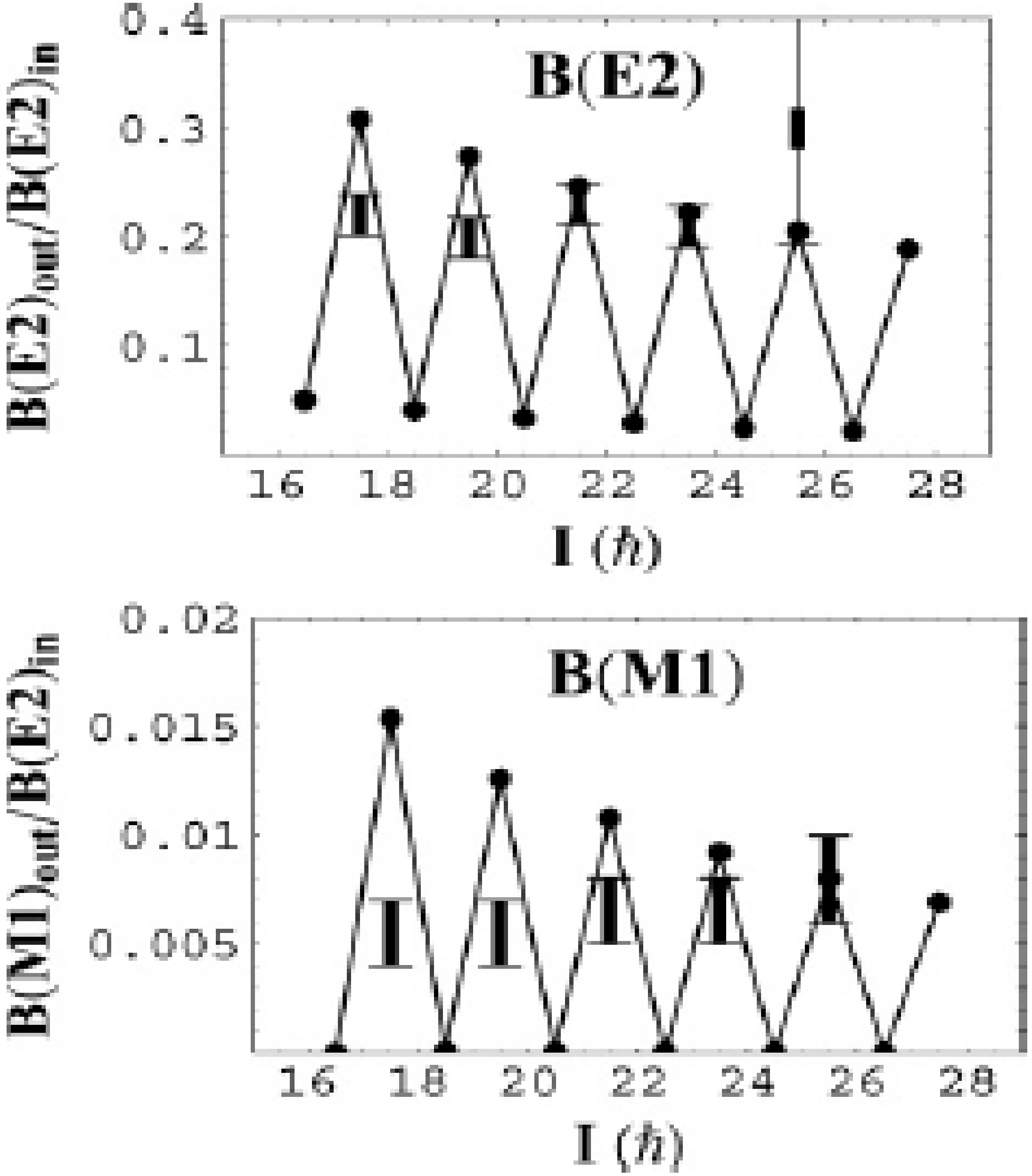


TABLE I. Comparison of the theoretical ratio $B(E2)_{out}/B(E2)_{in}$ with the experimental [9] one for ^{167}Ta, and three different initial angular momenta I. The theoretical prediction of $B(M1)_{out}/B(E2)_{in}$ is also given.

I	$B(E2)_{out}/B(E2)_{in}$ Expt.	$B(E2)_{out}/B(E2)_{in}$ Theory	$B(M1)_{out}/B(E2)_{in}$ Theory
39/2	0.37(4)	0.32	0.012
43/2	0.32(4)	0.29	0.011
47/2	0.36(4)	0.26	0.009

K.Sugawara-Tanabe and K.Tanabe, Phys.Rev.C82(2010)051303(R).



^{163}Lu

FIG. 8. A comparison between the theoretical and experimental ratios of $B(E2)_{out}/B(E2)_{in}$ (upper panel) and $B(M1)_{out}/B(E2)_{in}$ in units of $\mu_N^2/(eb)^2$ (lower panel) as functions of I in units of $\hbar$. The theoretical results with the rigid-body moments of inertia for the case of $s = 120$ and $\gamma = 17^\circ$ are represented by filled circles. The experimental data are from Ref. [1].

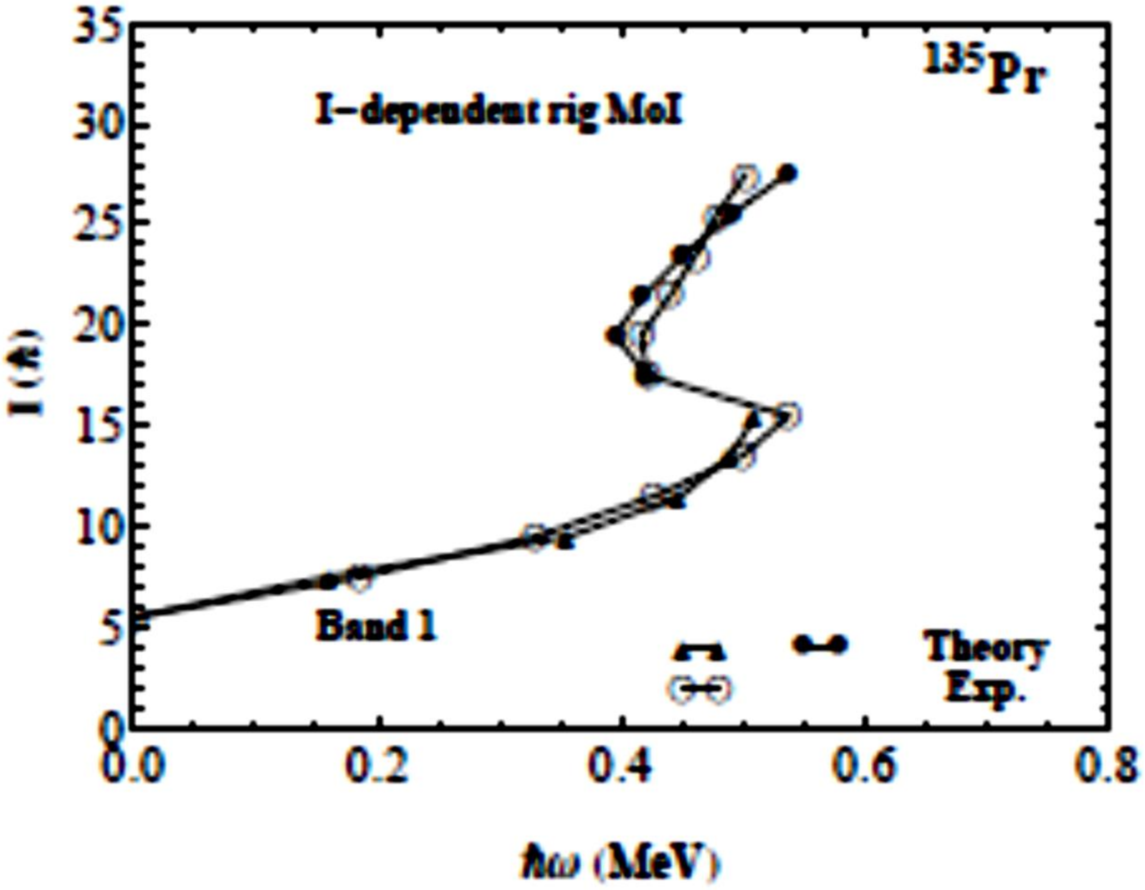


FIG. 19: Comparison between theory and experimental data in the backbending plot for Band 1. Theoretical values are shown by filled triangles for $I \leq 31/2$ and by filled circles for $I \geq 35/2$. Experimental values are shown by open circles. Filled circles are normalized at $I = 35/2$.

K.Tanabe and K.Sugawara-Tanabe,PRC95,064315(2017)

(1) As for the particle-rotor Hamiltonian with triaxially deformed rotor core interacting with single nuleon through Nilsson potential, diagonalization calculation within single-j shell model well describes the wobbling motion and related physical contents.

(2) Especially it reproduces well the electromagnetic transition rates like B(E2), B(M1)-values and E2-M1 mixing ratios, provided that increase of MoI with a.m. I is taken into account due to CAP effect.

(3) These suggest importance of microscopic formalism which describes change of MoI by taking account of pairing interaction and valence nucleons for odd A case .

Microscopic Formalism:

(1) Constrained-Hartree-Fock-Bogoliubov (CHFB) theory:

e.g. K.Tanabe and K.Sugawara-Tanabe,Phys.Lett.135B(1984)255. 158Er,160Yb

K.Sugawara-Tanabe and K.Tanabe,Phys.Lett.297B(1988)234. g-factors

K.Tanabe and K.Sugawara,Tanabe, Prog.Theor.Phys.83(1990)1148. 132Ce

K.Tanabe and K.Sugawara-Tanabe, Phys.Lett.B259B(1991)2. 132Ce, 134,136Nd

(2) HFB theory with quantum number projection:

e.g. K.Enami,K.Tanabe and N.Yoshinaga,Phys.Rev.C59(199)135.

Phys.Rev.C61(2000)027301. E2-transition

(3) Generator coordinate method:

e.g.K.Enami,K.Tanabe and N.Yoshinaga,Phys.Rev.C63(2001)044322.

Phys.Rev.C64(2001)044305, Phys.Rev.C65(2002)064308.

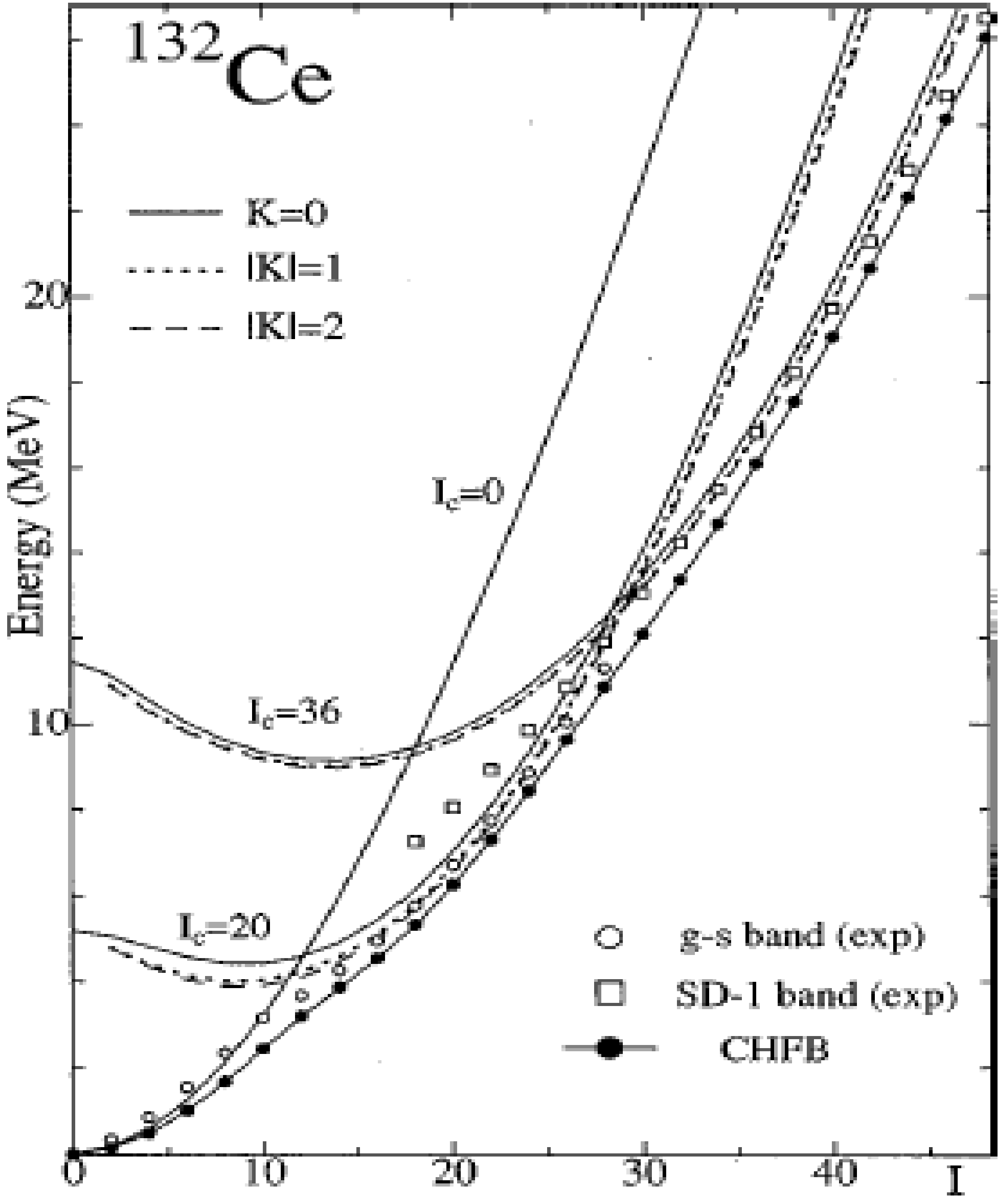


FIG. 10. Reproduction of three yrast bands for ^{132}Ce. The *g* band levels are calculated by angular momentum projection from the CHFB solution with $I_c=0$, the *s* band levels from the CHFB solution with $I_c=20$, and the superdeformed band levels from the CHFB solution with $I_c=36$. The CHFB solutions with given constraint spin $I=I_c$ are also plotted with solid circles connected by solid lines. The level energies along solid lines are calculated based on the physical space of $K=0$, those along dotted lines based on the space of $K=0,\pm1$, and those along dashed lines based on the space of $K=0,\pm1,\pm2$. Experimental data [23,24] are also plotted with open symbols as specified in the figure. In this plot, the excitation energy of the $I=18$ level in the SD band is assumed to be 7.25 MeV.

$$B(E2;I_\mathrm{i}\to I_\mathrm{f})=\frac{2I_\mathrm{f}+1}{2I_\mathrm{i}+1}|\langle\Psi_{I_\mathrm{f}}||\hat{Q}_2||\Psi_{I_\mathrm{i}}\rangle|^2, \qquad \langle\Psi_{I_\mathrm{f}}||\hat{Q}_2||\Psi_{I_\mathrm{i}}\rangle=\sum_{K,K',\nu}(I_\mathrm{i}K-\nu,2\nu|I_\mathrm{f}K)$$

$$\times\langle\Phi_\mathrm{f}|\hat{Q}_{2\nu}\hat{P}^{I_\mathrm{i}}_{K-\nu K'}|\Phi_\mathrm{i}\rangle F^{I_\mathrm{f}*}_{K}F^{I_\mathrm{i}}_{K'}$$

TABLE I. Force strengths employed in the calculations. Notations are similar to those in Table III in Ref. [10].

Nucleus	QQI interaction (MeV/b^4) [a]		MPI interaction (MeV)		QPI interaction (MeV/b^4) [a]
	$\chi_{\pi\pi}(=\chi_{\nu\nu})$	$\chi_{\pi\nu}$	$G^{(0)}_{\pi\pi}$	$G^{(0)}_{\nu\nu}$	$G^{(2)}$
^{160}Yb	0.0364	0.0888	0.2025	0.1492	$0.15G^{(0)}$
^{164}Yb	0.0361	0.0903	0.1946	0.1434	$0.15G^{(0)}$
^{168}Yb	0.0352	0.0915	0.1900	0.1400	$0.28G^{(0)}$

[a]Oscillator length $b=(\hbar/M\omega_0)^{1/2}(\hbar c/\mathrm{MeV})$.

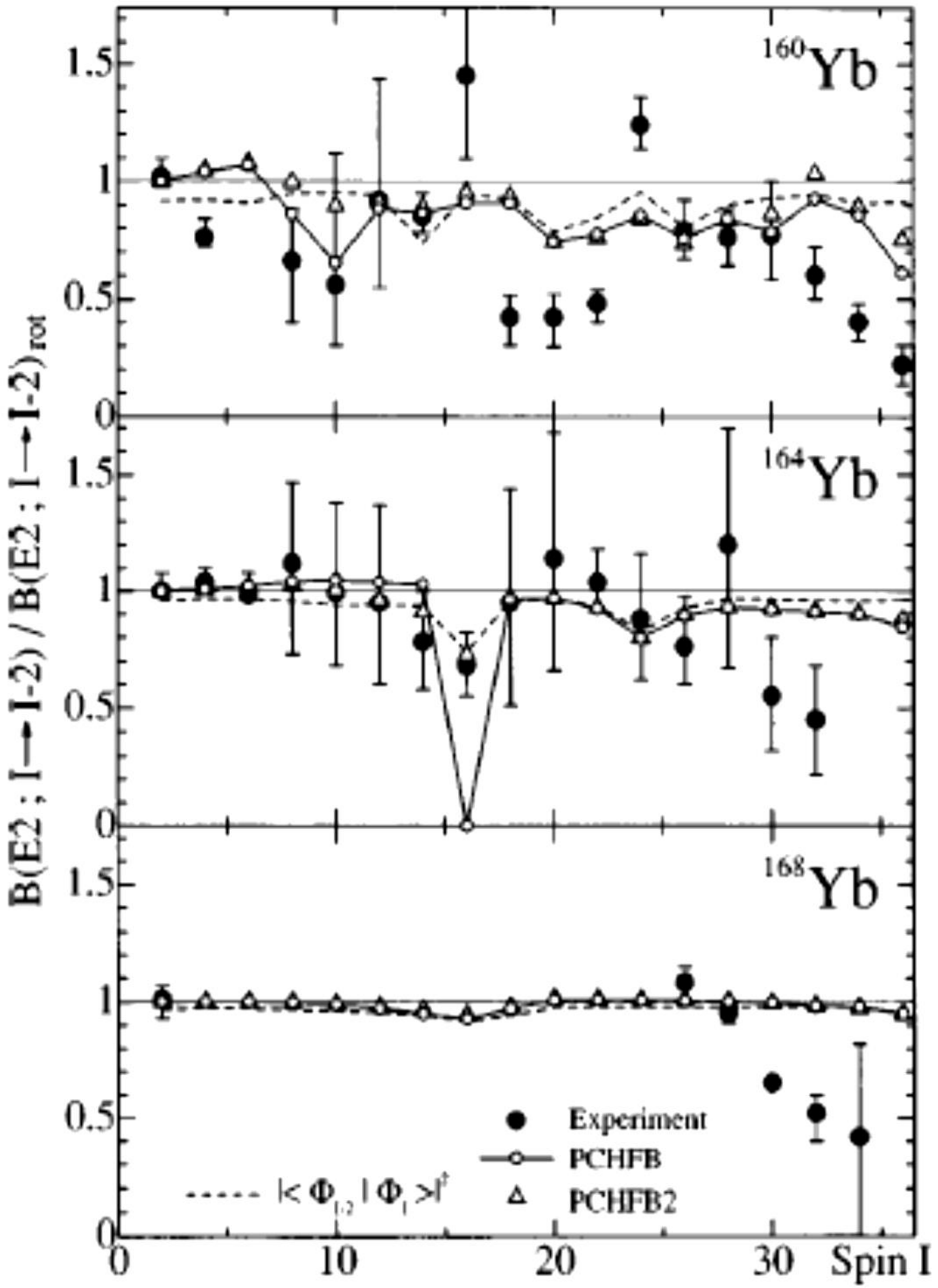


FIG. 1. Theoretical ratios $B(E2;I\to I-2)/B(E2;I\to I-2)_{rot}$ as functions of spin I calculated for ^{160,164,168}Yb. The theoretical results calculated from the angular momentum projection (PCHFB) are shown with open circles connected by solid lines, those employing the normalized projected wave function (PCHFB2) with open triangles and those from the CHFB overlaps ($|\langle\Phi_{I-2}|\Phi_I\rangle|^2$) with dashed lines. Experimental data are taken from the compilations in Ref. [17] based on the data in Refs. [16,17] for ^{160}Yb, Ref. [14] for ^{164}Yb, and Ref. [15] for ^{168}Yb.

K.Enami,K.Tanabe and N.Yoshinaga, PRC61,027301(1999).

Cranking model is certainly a method of calculating moment of inertia (MoI) within a microscopic formalism.

However, the cranking model works only in the special situation, where the rotational axis is confined to a certain spatial direction.

Our interest is a method how to extend the basic idea of the cranking model to the case of triaxial deformation.

Historically, in 1929, O. Klein recognized that **Euler equations** for the rotating rigid-body, i.e.,

$$\frac{dP}{dt} = \Big(\frac{1}{C} - \frac{1}{B}\Big)\Big(\frac{QR + RQ}{2}\Big), \quad \text{etc.} \tag{1}$$

can be derived from **the quantum-mechanical Heisenberg equation,**

$$i\frac{dP}{dt} = [P, H], \tag{2}$$

only when the quantum-mechanical angular momentum vector components (P, Q, R) of a rotating body satisfy **the commutation relations with negative signature like** $[P, Q] = -iR$**, etc.,**

Soon after, Casimir discussed the same problem with **direction cosines.**

Zur Frage der Quantelung des asymmetrischen Kreisels.

Von **O. Klein** in Kopenhagen.

(Eingegangen am 28. Oktober 1929.)

Matrizen auffassen. Die quantenmechanischen Bewegungsgleichungen für unser System werden dann lauten:

$$\left.\begin{aligned} \frac{dP}{dt} &= \frac{i}{h}(EP - PE), \qquad \frac{dQ}{dt} = \frac{i}{h}(EQ - QE), \\ \frac{dR}{dt} &= \frac{i}{h}(ER - RE), \end{aligned}\right\} \tag{2}$$

wo $i = \sqrt{-1}$ und h die durch 2π dividierte Plancksche Konstante bezeichnet. Um diese Gleichungen zusammen mit dem Energieausdruck (1) verwerten zu können, ist es notwendig, Vertauschungsrelationen für die Größen P, Q und R anzunehmen. Die folgenden Relationen

$$\left.\begin{aligned} ihP &= RQ - QR, \\ ihQ &= PR - RP, \\ ihR &= QP - PQ \end{aligned}\right\} \tag{3}$$

erfüllen die Korrespondenzforderung, daß sie zusammen mit den Gleichungen (2) zu Bewegungsgleichungen führen, die, soweit wir von dem Wirkungsquantum absehen können, mit den wohlbekannten Eulerschen Gleichungen für den asymmetrischen Kreisel übereinstimmen. In der Tat ergibt eine einfache Rechnung

$$\left.\begin{aligned} \frac{dP}{dt} &= \left(\frac{1}{C} - \frac{1}{B}\right)\left(\frac{QR + RQ}{2}\right), \qquad \frac{dQ}{dt} = \left(\frac{1}{A} - \frac{1}{C}\right)\left(\frac{RP + PR}{2}\right), \\ \frac{dR}{dt} &= \left(\frac{1}{B} - \frac{1}{A}\right)\left(\frac{PQ + QP}{2}\right) \end{aligned}\right\} \tag{4}$$

in möglichst naher Übereinstimmung mit den klassischen mechanischen Bewegungsgleichungen.

Zur quantenmechanischen Behandlung des Kreiselproblems.

Von **H. B. G. Casimir** in Kopenhagen.

(Eingegangen am 2. Dezember 1929.)

§ 1. Die Vertauschungsrelationen. Wir benutzen folgende Bezeichnungen:

P_1, P_2, P_3 Drehimpulskomponenten in bezug auf mitbewegte Achsen,
Q_1, Q_2, Q_3 Drehimpulskomponenten in bezug auf ruhende Achsen,
D_{ik} Komponenten des i-ten Einheitsvektors im mitbewegten System in Richtung der k-ten festen Achse,
A_1, A_2, A_3 Trägheitsmomente, bezogen auf das körperfeste System,
H Energie $= \frac{1}{2}\sum_i \frac{P_i^2}{A_i}$.

und den Kleinschen

$$P_1 P_2 - P_2 P_1 = -ihP_3$$

noch die folgenden Vertauschungsregeln annehmen:

$$\begin{aligned} Q_i P_k - P_k Q_i &= 0, \\ P_1 D_{2k} - D_{2k} P_1 &= -ihD_{3k}, \\ P_2 D_{1k} - D_{1k} P_2 &= +ihD_{3k}, \\ P_1 D_{1k} - D_{1k} P_1 &= 0. \end{aligned}$$

Angular momentum operators describing the rotational motion of a body.
The space-fixed components of the angular momentum operator **J** are given by

$$\begin{pmatrix} J_X \\ J_Y \\ J_Z \end{pmatrix} = -i \begin{pmatrix} -\cos\alpha\cot\beta & -\sin\alpha & \cos\alpha/\sin\beta \\ -\sin\alpha\cot\beta & \cos\alpha & \sin\alpha/\sin\beta \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{pmatrix} \tag{3}$$

These space-fixed components satisfy the commutation relations like

$$[J_X, J_Y] = iJ_Z. \tag{4}$$

The body-fixed components of the angular momentum operator **J** are represented by

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = -i \begin{pmatrix} -\cos\gamma/\sin\beta & \sin\gamma & \cos\gamma\cot\beta \\ \sin\gamma/\sin\beta & \cos\gamma & -\sin\gamma\cot\beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{pmatrix} \tag{5}$$

These space-fixed components satisfy the commutation relations like

$$[J_x, J_y] = -iJ_z. \tag{6}$$

These body-fixed components satisfy the commutation relations like

$$[J_x, J_y] = -iJ_z. \tag{6}$$

Angular momentum components in two coordinate systems are related each other as

$$J_k = \sum_{L=X,Y,Z} \lambda_{kL} J_L \tag{7}$$

with the set of direction cosines λ_{kL} ($L = X, Y, Z$ and $k = x, y, z$), i.e.,

$$\begin{aligned}(\lambda_{kL}) &= [D^1(\alpha,\beta,\gamma)^{tr}]_{kL} = D^1(-\gamma,-\beta,-\alpha)_{kL} \\ &= \begin{pmatrix} \cos\beta\cos\gamma\cos\alpha - \sin\gamma\sin\alpha & \cos\beta\cos\gamma\sin\alpha + \sin\gamma\cos\alpha & -\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma\cos\alpha - \cos\gamma\sin\alpha & -\cos\beta\sin\gamma\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\gamma \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix},\end{aligned} \tag{8}$$

where $D^1(\alpha,\beta,\gamma)$ is Wigner D-function in the spin-1 representation.

Useful commutation relatins among angular momentum components and direction cosines:

·Angular monentum vector of a valence nucleon (A-th nucleon):

$$\mathbf{j}^A \qquad (j_x^A, j_y^A, j_z^A), \qquad (j_X^A, j_Y^A, j_Z^A)$$

·Total angular momentum of the other $A-1$ nucleons :

$$\mathbf{R} \equiv \sum_{\mu=1}^{A-1} \mathbf{j}^\mu \quad (R_x, R_y, R_z), \quad (R_X, R_Y, R_Z)$$

·Total angular momentum of A nucleons:

$$\mathbf{I} \equiv \mathbf{R} + \mathbf{j}^A \quad (I_x, I_y, I_z), \quad (I_X, I_Y, I_Z)$$

·Useful commutation relations:

$$[I_k, I_l] = -iI_{k\times l}, \quad [R_k, R_l] = -iR_{k\times l},$$

$$[I_k, \lambda_{lK}] = -i\lambda_{k\times lK}, \quad [R_k, \lambda_{lK}] = -i\lambda_{k\times lK}; \qquad [I_k, j_l^{\mathrm{A}}] = 0.$$

$$[I_K, I_L] = iI_{K\times L}, \quad [R_K, R_L] = iR_{K\times L}, \quad [j_K, j_L] = ij_{K\times L}, \quad [R_K, j_L^{\mathrm{A}}] = 0$$

$$[I_K, \lambda_{kL}] = i\lambda_{kK\times L}, \quad [R_K, \lambda_{kL}] = i\lambda_{kK\times L}.$$

Hamiltonian in the rotating frame is given by

$$H^{\rm rot} = H - \sum_{k=x,y,z} \omega_k I_k, \tag{9}$$

where the angular frequencies and the angular momentum components are not necessarily be along the principal axes of the rotating body. Only when these coincide with the principal axes of the body, the angular frequencies can be expressed in tems of the moments of inertias (MoI) $\mathcal{J}_k$ as $\omega_k = I_k/\mathcal{J}_k$. Then, we have for the system with one valence nucleon with its angular momentum $\mathbf{j}^A$

$$H^{\rm rot} = H - \sum_{k=x,y,z} \frac{(I_k - j_k^A)I_k}{\mathcal{J}_k}. \tag{10}$$

For a pure rotor case with $H = \sum_{k=x,y,z}(I_k - j_k^A)^2/(2\mathcal{J}_k)$, we have

$$H^{\rm rot} = -\sum_{k=x,y,z} \frac{I_k^2 - (j_k^A)^2}{2\mathcal{J}_k}. \tag{11}$$

Therefore, when $I_k^2 > (j_k^A)^2$ for all $k = x, y, z$, $H^{\rm rot}$ becomes negative, so that it is larger for larger values of MoI. In case of variational calculation, we have to look for the maximum of $H^{\rm rot}$ rather than its minimum.

Regarding rotational frequencies $\omega_k = (I_k - j_k^{\mathbf{A}})/\mathcal{J}_k$ as c-numbers, and an additional term $H' \equiv -\sum_{k=x,y,z} \omega_k I_k$ as a perturbation to H in $H^{\rm rot}$.
We start from the complete set of the solution to H as

$$Hu_n = E_n u_n, \quad \{(u_n, E_n), n = 0, 1, 2, \cdots\}. \tag{12}$$

and the expectation value of an physical quantity $\hat{O}$ is calcualated in the lowest order approximation as

$$\begin{aligned} <\psi|\hat{O}|\psi> &= <u_m|\hat{O}|u_m> \\ &+ \sum_{k(\neq m)} \left[\frac{<u_m|\hat{O}|u_k><u_k|H'|u_m>}{E_m - E_k} + \frac{<u_m|H'|u_k><u_k|\hat{O}|u_m>}{E_m - E_k} \right]. \end{aligned} \tag{13}$$

Regarding u_m as the ground state u_0, and identifying $\hat{O}$ with $I_k - j_k^{\mathbf{A}}$ for $k = x, y, z$, alternatively, we get a result which can be expressed in a matrix form like

$$\begin{pmatrix} <\psi|(I - j^{\mathbf{A}})_x|\psi> - <u_0|(I - j^{\mathbf{A}})_x|u_0> \\ <\psi|(I - j^{\mathbf{A}})_y|\psi> - <u_0|(I - j^{\mathbf{A}})_y|u_0> \\ <\psi|(I - j^{\mathbf{A}})_z|\psi> - <u_0|(I - j^{\mathbf{A}})_z|u_0> \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xx} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}. \tag{14}$$

$$
\begin{pmatrix} I_x \\ I_y \\ I_x \end{pmatrix}
= \begin{pmatrix} \cos\beta\cos\gamma\cos\alpha - \sin\gamma\sin\alpha & \cos\beta\cos\gamma\sin\alpha + \sin\gamma\cos\alpha & -\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma\cos\alpha - \cos\gamma\sin\alpha & -\cos\beta\sin\gamma\sin\alpha + \cos\gamma\cos\alpha & \sin\beta\sin\gamma \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix} \begin{pmatrix} I_X \\ I_Y \\ I_Z \end{pmatrix}
$$
$$
= \begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-i\alpha}(\cos\beta\cos\gamma - i\sin\gamma) & -\sin\beta\cos\gamma & \frac{1}{\sqrt{2}}e^{i\alpha}(\cos\beta\cos\gamma + i\sin\gamma) \\ \frac{1}{\sqrt{2}}e^{-i\alpha}(\cos\beta\sin\gamma + i\cos\gamma) & \sin\beta\sin\gamma & -\frac{1}{\sqrt{2}}e^{i\alpha}(\cos\beta\sin\gamma - i\cos\gamma) \\ -\frac{1}{\sqrt{2}}e^{-i\alpha}\sin\beta & \cos\beta & \frac{1}{\sqrt{2}}e^{i\alpha}\sin\beta \end{pmatrix} \begin{pmatrix} I_{+1} \\ I_0 \\ I_{-1} \end{pmatrix}, \tag{15}
$$

where

$$
I_{+1} = -\frac{1}{\sqrt{2}}(I_X + iI_Y), \qquad I_0 = I_Z, \qquad I_{-1} = \frac{1}{\sqrt{2}}(I_X - iI_Y). \tag{16}
$$

Similar relatins hold also between the body-fixed components $j^{\mu}_{x,y,z}$ and the space-fixed ones $j^{\mu}_{X,Y,Z}$ of the single-particle angular momenta $\mathbf{j}^{\mu}$ of each nucleons numbered by $\mu = 1, 2, \cdots, A$.

Thus, we derive a concise expression for the inertial tensor in a real symmetric 3×3 matrix form as

$$A_{kl} = AM_{kl}\,,\ (k,l = x,y,z) \tag{17}$$

with

$$(M_{kl}) \equiv \begin{pmatrix} \cos^2\beta\cos^2\gamma+\sin^2\gamma & \sin^2\beta\cos\gamma\sin\gamma & \cos\beta\sin\beta\cos\gamma \\ \sin^2\beta\cos\gamma\sin\gamma & \cos^2\beta\sin^2\gamma+\cos^2\gamma & \cos\beta\sin\beta\sin\gamma \\ \cos\beta\sin\beta\cos\gamma & \cos\beta\sin\beta\sin\gamma & \sin^2\beta \end{pmatrix} \tag{18}$$

and

$$A = \sum_{\mu=1}^{A-1}\Big[\sum_{f_1{}^\mu > i^\mu}\frac{|<f_1{}^\mu|I_{+1}|i^\mu>|^2}{\epsilon_{f_1}{}^\mu - \epsilon_i{}^\mu} + \sum_{f_2{}^\mu > i^\mu}\frac{|<f_1{}^\mu|I_{+1}|i^\mu>|^2}{\epsilon_{f_2}{}^\mu - \epsilon_i{}^\mu}\Big]. \tag{19}$$

In case of the BCS model, this A value should be replaced by

$$A_{\rm Belyaev} = 2\sum\nolimits'_{k>k'}\frac{|<k|I_X|k'>|^2}{E_k+E'_k}(u_k v_{k'} - v_k u_{k'})^2. \tag{20}$$

In order to diagonalize the symmeric matrix (A_{kl}), first, we determine the direction of the body-fixed frame (β, γ) by requiring that two matrix elements from three nondiagonal elements (M_{xy}, M_{xy}, M_{xy}) vanish. The eigenvalues of the matrix M are turned out to be $1, 1$ and 0 independent of β. Even if we put $\beta = 0$, we again get the same set of eigenvalues independent of γ.

Thus, the system always has two MoI with common value in two orthogonal directions, while the MoI is vanishing in the third orthogonal direction. Therefore, we do not find any triaxial deformation within the framework of this extended cranking model with an approximation with c-number angular frequencies ω's.

[Second Attempt]

In order to take into account (1) the effect caused by the valence nucleon, and (2) direct treatment of the original angular momentum-dependent perturbation without employing c-number angular frequency, we develop the other approximation method in what follows.

We start with the original form of the perturbation due to rotational effect as

$$H' = -\sum_{k=x,y,z} \frac{(I_k - j_k^{\mathrm{A}})I_k}{\mathcal{J}_k} \tag{21}$$

Thus, the perturbation formula gives

$$< \psi|I_p|\psi > - < u_0|I_p|u_0 >= \sum_{q=x,y,z} \sum_{k(\neq 0)} \frac{< u_0|I_p|u_k >< u_k|I_q(I_q - j_q^{\mathrm{A}})|u_0 > +h.c.}{(E_k - E_0)\mathcal{J}_q}, \tag{22}$$

for $p = x, y, z$.

Making use of direction cosines $\{\lambda_{kL}\}$, we express the r.h.s. of the formula in terms of the space-fixed components of angualar momenta, and rewrite the whole expression with operators (I_{+1}, I_0, I_{-1}) and (j_{+1}, j_0, j_{-1}) for practical calculations. For simplicity, expressing $|u_n>$ as $|n>$, we arrive at

$$< \psi|I_p|\psi > - < 0|I_p|0 >= \sum_{q=x,y,z} M_{pq}/\mathcal{J}_q, \quad M_{pq} = \sum_{n\neq 0} N(n)_{pq}/(E_n - E_0) \qquad (23)$$

with

$$N(n)_{pq} \equiv < 0|\sum_{\mu=1}^{A} j_p^{\mu}|n >< n|\sum_{\rho=1}^{A} j_q^{\rho} \sum_{\nu=1}^{A-1} j_q^{\nu}|0 > +h.c.. \qquad (24)$$

However, this scheme still fails to take into account enough correlations to lead triaxial deformation.

[Third Attempt]

(1) Quantum mechanically, the expectation value of ang.mom. operator in linear order is not preferable.

(2) It can be expected that more correlations will be taken into account by the expectation value $\langle I_p^2 \rangle$ rather than $\langle I_p \rangle$ as in the previous cases.

In this model, we solve a set of the algebraic equation given by

$$< \psi|I_p^2|\psi > - < 0|I_p^2|0 >= \sum_{q=x,y,z} M_{pq}/\mathcal{J}_q, \quad M_{pq} = \sum_{n\neq 0} N_{pq}(n)/(E_n - E_0) \tag{25}$$

with

$$N_{pq}(n) =< 0|(\sum_{\mu=1}^{\mathbf{A}} j_p^\mu)^2|n >< n| \sum_{\rho=1}^{\mathbf{A}} j_q^\rho \sum_{\sigma=1}^{\mathbf{A}-1} j_q^\sigma|0 > +h.c.. \tag{26}$$

In the caculation, we fully use the commutation relations among a.m. components and direction cosines as follows, e.g.,

$$\begin{aligned} (I_p)^2 &= \sum_{P,Q} (\lambda_{pP} I_P)(\lambda_{pQ} I_Q) \\ &= \sum_{P,Q} \lambda_{pP}(\lambda_{pQ} I_P + i\lambda_{pP\times Q}) I_Q, \\ (I_p)^2 &= \sum_{P,Q} I_P(I_Q\lambda_{pP} - i\lambda_{pP\times Q})\lambda_{pQ}. \end{aligned} \tag{27}$$

In practice of calculation with this scheme, we use cranking assumption, i.e., $<\psi|(I_p)^2|\psi> - <0|(I_p)^2|0> = I(I+1) - I_0(I_0+1)$ for $p = x, y$ and z, alternatively. In the 3×3 matrix M_{pq}, we determine two Eulerian angles (β, γ) by requiring that two non-diagonal matrix elements from three non-daigonal elements (M_{xy}, M_{yz}, M_{zx}) vanish. For example, when $M_{xy} = M_{zx} = 0$, then the the one MoI $\mathcal{J}_x$ is already determined, and the reamaining two, i.e., $\mathcal{J}_y$ and $\mathcal{J}_z$, can be soleved from the remaining two relations in Eq.(25).

In conclusion, I would stress importance of microscopic theory to describe nuclear deformation mechanism:

Core polarization due to valence nucleon ?

We have to answer, Rigid-body , or hydrodynamical ?